PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1978 **288**, 611-625
doi: 10.1098/rsta.1978.0037

Phil. Trans. R. Soc. Lond. A. **288**, 611–625 (1978) [611]
Printed in Great Britain

On the cooling of ascending andesitic magma

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In ascending through the lithosphere, andesitic magma probably cools by about 300 K. Since the ascent velocity and dynamics of ascent are unknown, several different phenomenological cooling models are considered to account for this temperature decrease. These results are compared to the phase relations of andesitic magma in order to estimate an ascent velocity which can be used to investigate a dynamic model of ascent. The cooling models approximate an initially crystal-free magma ascending by elastic crack propagation (plate model), viscous blobs (spherical model), and by flow up a pipe. It is shown that heat transfer is predominantly by convection and conduction, and that an adiabatic ascent is unlikely. The calculated cooling curves have the general shape of the geotherm and are concave to the liquidus and solidus of the magma. Hence the magma is likely to become superheated over much of its ascent which precludes crystal fractionation, and the petrology of the lavas seems to support this. For a viscous sphere of magma the ascent velocity must be greater than about 10^{-7} m s^{-1}, but for a crack of the same volume, because of its large surface area, ascent is at least 10^{-3} m s^{-1}. Because of the paucity of mantle xenoliths in andesitic lavas, this latter ascent velocity seems unlikely.

Introduction

On its way to the surface, andesitic magma must traverse the lithosphere, a region where the Earth's temperature changes by more than 1000 K. Magma ascending infinitely slowly will maintain thermal equilibrium with the lithosphere and solidify at depth. A rapid ascent may be adiabatic and, providing the magma initially contains few crystals, it will appear superheated at the surface. The final magma temperature will in general be proportional to the ascent velocity and to the temperature differential between the magma and the local lithosphere; the rate of cooling will increase greatly as the surface is approached. For magma ascending through the lithosphere a family of cooling curves can be calculated as a function of ascent velocity without a detailed knowledge of the dynamics of ascent. Since the magma must arrive without solidifying, the magma liquidus–solidus relations can be used to estimate a range of probable ascent velocities. Once a range of ascent velocities is estimated the conditions under which a dynamic model satisfies this requirement can be explored.

These models are intended to describe magma ascent beneath an island arc, specifically the Aleutian arc where the tectonics and geography of volcanism are relatively well known. Volcanism occurs here at discrete, fairly evenly spaced (*ca.* 70 km) volcanic centres along the arc. Two volcanoes, Bogoslof and Amak, lie 50 km north of the main arc, but the volcanism is in general restricted to a narrow (*ca.* 10 km) volcanic front where the magma is sporadically emitted from the same locations for the lifetime of the arc. Once the spatial pattern of volcanoes is established it remains relatively fixed; over millions of years magma bodies travel the same path through the lithosphere. This general path must become increasingly warm thereby insulating the magma and allowing it to rise more slowly without solidifying.

Although these models do not depend critically on a specific means of magma generation, when an example is cited it is drawn from the Benioff zone magmatism model (Marsh 1973;

Marsh & Carmichael 1974; Marsh 1976*a, b, c*). Magma is primarily produced through partial melting of subducted oceanic crust, quartz–eclogite, at a depth of 100–150 km and a temperature of 1350–1450 °C. In the near-surface environment a typical magma may contain about 1 % (by mass) or less of water. Its first phenocrysts, anorthitic plagioclase, begin crystallizing at about 1200–1250 °C at perhaps 3–7 kbar† total pressure, and the eruption temperature is about 1100 °C. The spatial distribution of volcanic centres can be interpreted to arise from the gravitational fluid instability of andesitic magma in the region of the Benioff zone. In traversing the lithosphere the magma probably cools by 200–300 °C, and cooling models which account for this temperature drop are sought to estimate an ascent velocity.

Heat transfer

Adiabatic ascent

From an ascending body of magma heat is conducted across the magma–wall rock interface and then convected away by the wall rock. If the wall rock is a perfect insulator the magma will not cool by conduction or convection but only adiabatically. Since rock is not a perfect insulator, for an adiabatic ascent the heat lost to the wall rock must be replaced exactly by heat production within the magma. This possibility can be evaluated by expressing the previous statement in the form (Batchelor 1967, p. 156):

$$\rho T\left(\frac{\mathrm{D}S}{\mathrm{D}t}\right) = C_p\left(\frac{\mathrm{D}T}{\mathrm{D}t}\right) - \alpha T\left(\frac{\mathrm{D}P_0}{\mathrm{D}t}\right) = \lambda\Delta^2 + \frac{\partial}{\partial X_i}\left(K_c\frac{\partial T}{\partial X_i}\right) + e = 0, \tag{1}$$

which describes the change in entropy (S) in a single-phase compressible fluid, where $\mathrm{D}/\mathrm{D}t$ is the material derivative, ρ is density, T is temperature, t is time, V is velocity, C_p is specific heat at constant pressure, X is a spatial coordinate, λ is the coefficient of viscous expansion, α is thermal expansion, P_0 is the hydrostatic pressure, Δ is the trace of the dilational tensor, K_c is thermal conductivity, and e describes any source of entropy (e.g. crystallization, viscous dissipation). Reversible cooling due to adiabatic expansion is described by the second and third terms; for an andesitic liquid this amounts to about 60 °C per 100 km. The fourth term describes an irreversible addition of energy to the fluid through frictional effects during expansion, this effect is difficult to evaluate since λ is unknown, but it is generally small (Batchelor 1967) and it is henceforth neglected. To satisfy (1) the last two terms must be numerically equal and of opposite sign. This possibility is now evaluated.

Viscous dissipation due to internal convection is a likely heat source; here e is proportional to $\mu V^2/d^2$ and the loss of heat by conduction is proportional to $K_c\Delta T/d^2$. The importance of e relative to conduction is given by

$$\mu V^2/K_c\Delta T, \tag{2}$$

which is the so-called Brinkman number, where μ is viscosity, and ΔT is the temperature difference which drives conduction and convection over the distance d. For $\mu = 10^{-1}\,K_g\,\mathrm{m^{-1}\,s^{-1}}$, $V = 10^{-1}\,\mathrm{m\,s^{-1}}$ (as observed in Hawaiian lava lakes), $K_c = 4\,K_g\,\mathrm{m\,s^{-3}\,K^{-1}}$, and $\Delta T = 10$, (2) is about 10^{-4}. An even more general and startling measure of this imbalance is found by using the results of Hewitt, McKenzie & Weiss (1975). Thus conductive losses are greatly more important than the heat gained through viscous dissipation. If instead of viscous dissipation e is assumed to

† 1 kbar = 10^8 Pa.

be heat produced by crystallization it is clear that, in order for crystallization to proceed, the magma must cool (neglecting H_2O-saturated decompressional crystallization) and hence $e/(K_c \Delta T/d^2)$ must again be less than unity. Crystallization during ascent is a nonadiabatic effect.

Adiabatic ascent of magma seems an unlikely occurrence. This is consistent with Lang's (1972) result which showed an adiabatic ascent to be nearly isothermal. This would produce a superheated lava which is not common (i.e. no phenocrysts) among andesites. The remainder of this paper assesses magmatic cooling due to convection and conduction of heat by lithospheric wall rock during ascent.

Non-adiabatic ascent

In addition to convective–conductive heat transfer and adiabatic cooling mentioned already, magmatic cooling may also be influenced by volatile loss, fusion of wall rock, interfacial chemical reaction, and viscous dissipation in the wall rock. The effect of these sources on any cooling model is summarized in the following equation describing the thermal state of magma with approximately constant thermal diffusivity (K), specific heat (C_p), and density (ρ):

$$\frac{\partial T}{\partial t} + V_i \frac{\partial T}{\partial X_i} = K \frac{\partial^2 T}{\partial X_i^2} + e/\rho C_p; \tag{3}$$

the heat sources are described by e. To judge the relative importance of the terms in (3) it is convenient to introduce the dimensionless variables $T' = T/\Delta T_0$, $t' = Kt/a^2$, $\partial'/\partial X_i = a\,\partial/\partial X_i$ and $V' = V_i/V_0$. Because e can take on several different forms it is left in its primitive form. Upon substitution and rearranging (3) becomes

$$\frac{\partial T'}{\partial t'} + Pe V_i' \left(\frac{\partial T'}{\partial X_i'}\right) = \frac{\partial^2 T'}{\partial X_i'^2} + \frac{ea^2}{K_c \Delta T_0}; \tag{4}$$

Pe is the Peclet number ($V_0 a/K$), V_0 is a characteristic ascent velocity, ΔT_0 is a typical temperature difference between the magma and the wall rock, and a is a characteristic size (radius) of the magma body. These scales are chosen to make the magnitude of each primed quantity (i.e. the derivatives) of order unity. The importance of convective relative to conductive cooling is measured by Pe, and the importance of heat sources relative to heat transfer is given by the ratio of the last term to Pe.

To anticipate later results, for a rather slow ascent velocity of $10^{-7}\,\mathrm{m\,s^{-1}}$, $a = 1$ km, and $K = 10^{-6}\,\mathrm{m^2\,s^{-1}}$, Pe is found to be 100. Convective cooling is at least one hundred times more important than conduction.

If the magma saturates with H_2O the heat taken for exsolution will cool the magma (Burnham & Davis 1974) as will subsequent escape of the exsolved fluid into the wall rock. To evaluate this heat sink, detailed knowledge of bubble nucleation and migration and wall rock permeability must be available. Hori (1964), in considering the effect of convective cooling by ground-water to be like fluid flow from an intrusive, found that convective cooling would be about as important as conductive cooling. That is, the last term in (4) is approximately unity; however, since Pe is large this effect is less important than convective–conductive cooling by the moving wall rock itself. Moreover, since andesitic magma appears to be undersaturated with H_2O even in the near-surface environment (Marsh 1976*a*; Ewart 1976), this means of cooling is for the most part unimportant until the body is very near the surface. As the magma nears the surface its ascent velocity may decrease whence, with certain values of wall rock permeability, this effect could become important. An approximate method of treatment of this matter is given by Jaeger (1964).

At the magma–wall rock interface chemical reaction between the magma and the wall rock

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may affect heat transfer from the magma by providing a source or sink for heat. For a first-order chemical reaction taking place on the surface of a *solid* sphere, Chen & Pfeffer (1970) found heat transfer to be enhanced – or hindered, this depending on the sign of the enthalpy change – by $2B/Pe^{\frac{1}{3}}$, where B^2 is the last term in (4). Judging from the results of Levich (1962), for a *liquid* sphere (i.e. magma) this may become $2B/Pe^{\frac{1}{2}}$. To determine B, however, the reaction rate constant must be available, but since this is generally unknown this effect cannot be evaluated. It is probably not important.

The effect of wall rock fusion on heat transfer can be estimated qualitatively. If during the wall rock–magma interaction the wall rock temperature increases by, say, ΔT, without fusion the magma must supply energy amounting to $\rho C_p \Delta T$, while with fusion the necessary energy is approximately $\rho C_p \Delta T + \rho L \Delta T/(T_L - T_s)$; L is the heat of fusion, T_L is the wall rock liquidus temperature, and T_s is its solidus temperature and each is at the appropriate pressure. The effect of wall rock fusion is to raise the apparent heat capacity of the wall rock which lowers its thermal diffusivity ($K = K_c/\rho C_p$) – otherwise considered approximately constant – and in turn increases Pe. For reasonable values of L and $(T_L - T_s)$ the apparent heat capacity, and hence Pe, is doubled. As shall be shown later, doubling Pe increases the overall heat transfer from the magma by $\sqrt{2}$; this effect is not significant. More qualitatively, raising the apparent wall rock heat capacity through fusion enables the same amount of heat to be carried away by a smaller volume of wall rock. With wall rock fusion the magma must supply twice as much energy as is needed without fusion to raise the wall rock temperature by the same amount.

The presence of crystals within the magma can supply heat through crystallization or absorb heat when they undergo fusion. During magmatic crystallization the heat evolved allows the magma to cool more slowly. Yet even when there is only conductive cooling of the magma by the wall rock (i.e. when the ascent velocity is zero) the magma temperature must still drop with time to sustain crystallization; thus the last term in (4) must be less than unity. When the magma is ascending, Pe can become large and, providing it is sufficiently large, the additional heat of crystallization will hardly affect magmatic cooling (i.e. $(ea^2/K_c \Delta T)/Pe \ll 1$).

Crystals initially in the magma when the magma begins ascent which are later melted use up energy during fusion and hence they have the effect of buffering the magma temperature. This effect can seriously affect magmatic cooling, as shall be shown later, yet it is difficult to treat. So, instead, for the present a standard state is chosen to be a magma which is initially crystal-free, and this effect will be discussed again later.

The energy available for viscous dissipation during wall rock deformation is bounded by the rate of loss of potential energy by the magma during ascent, $e \leqslant \Delta\rho g V$, where $\Delta\rho$ is the density difference between magma and wall rock. The last term in (4) then becomes $\Delta\rho g V a^2/K_c \Delta T$. As defined earlier, a is a typical length scale associated with the magma; here, however, for an accurate evaluation of this quantity a must be taken as the thickness of the thermal halo about the magma. For all but the slowest ascent velocities (i.e. $V \sim 10^{-8}\,\mathrm{m\,s^{-1}}$) this length is about one tenth or less of the body radius. With $\Delta\rho = 0.5 \times 10^3\,\mathrm{kg\,m^{-3}}$, $g = 10\,\mathrm{m\,s^{-2}}$, $K_c = 4.1\,\mathrm{W\,m^{-1}\,deg^{-1}}$, $\Delta T = 10^2\,\mathrm{K}$, and to anticipate later results $V = 10^{-5}\,\mathrm{m\,s^{-1}}$, and $a = 10^2\,\mathrm{m}$, this quantity is about 1. Yet since Pe is thought to be large (greater than about 100) magmatic cooling will be relatively unaffected by wall rock heating due to viscous dissipation during deformation; hence this effect is also neglected in the cooling model.

In conclusion, the sum of the processes contributing to e in (4) is probably much less important than magmatic cooling due to convective–conductive heat transfer away from the magma by the moving wall rock (i.e. $ea^2/K_c\Delta T)/Pe \ll 1$). This is only true for a moving body of magma. Once the body stops crystallization becomes important. If the body moves sufficiently fast heat production in the wall rock due to viscous dissipation accompanying deformation may hinder cooling. These conclusions hold principally for magma ascent by viscous deformation of the wall rock.

Critical parameters

Body shape and size are the most important parameters in formulating a heat transfer model. Any solution to the energy equation (4) at small Pe will be proportional to the dimensionless group Kt/a^2 (Fourier parameter), while at large Pe it is proportional to Pe. The size of the body is characterized by a and, since it is contained in these dimensionless groups, initially it need not be directly considered. In the end, however, to extract an actual ascent velocity a judicious choice of size is necessary. This is discussed later.

The shape of an ascending magma depends on its mode of transfer. Transmission by elastic crack propagation will produce a dike-shaped body where the aspect ratio (length/width) is about 2500 (Weertman 1971 *a*, *b*; Takeuchi & Kikuchi 1973; Pollard & Muller 1976). Here a thin plate or a strongly flattened ellipsoid adequately approximates the body shape.

Judging from the nature of plutons in the Aleutian Islands, diapiric rise of viscous blobs of magma represents another likely mode of magma transmission through the lithosphere. At low fluid velocities (small Reynolds number) inertial effects are negligible in deforming a blob of fluid from the shape of a sphere (Batchelor 1967), hence the shape of magma rising as a viscous diapir can be safely approximated as a sphere. An ellipsoidal form could just as well be used; the results for any roughly spherical body are hardly different.

The experiments by Grout (1945) which are commonly quoted as evidence that rising blobs of magma may be roughly spherical suffer from the effects of surface tension, and are on this aspect unrealistic.

Formulation and calculations

Regardless of the exact mode of magma transfer a phenomenological model which accurately describes its cooling can be constructed without knowledge of the intricacies of the dynamics of ascent.

Viscous sphere

A large blob of magma is imagined to rise through the lithosphere by softening a thin rind of wall rock causing it to flow past the magma. Heat conducts from the thermally well mixed magma across its border and is carried away by the moving wall rock. The situation is similar to monitoring heat loss from a hot drop of fluid falling through a cooler fluid; this has been widely investigated for industrial purposes (e.g. Kronig & Brink 1951; Bird, Stewart & Lightfoot 1960, p. 409; Levich 1962, p. 404; Head & Hellums 1966). An analytical solution essentially involves finding the relation between the temperature field and velocity field both inside and external to the magma. Since a complete analytical solution to this problem has yet to be discovered, a phenomenological approach is preferred.

The total heat flux (Q) leaving a globe of magma is proportional to the product of its surface area and the temperature difference between the magma (T) and the mantle wall rock far from the magma ($T_m(t)$); the wall rock temperature diminishes in traversing the lithosphere and hence it is a function of time. The magma temperature and all physical properties are taken to be averages.

$$Q \sim 4\pi a^2(T - T_m(t)). \tag{5}$$

An equality is made by defining a heat transfer coefficient h containing all the complicated effects of the thermo-mechanical interaction between the moving magma and the wall rock. Then, (6) is a definition of h.

$$Q = 4\pi a^2 h(T - T_m(t)). \tag{6}$$

It is convenient to express h in terms of the dimensionless group ah/K_c, which is the so-called Nusselt number, Nu. For forced convection, which is the present concern, one can show Nu to be a function of Pe on dimensional grounds (e.g. Bird *et al.* 1960, chapter 13). Equating Q to the time rate of change of magma internal energy,

$$\mathrm{d}E/\mathrm{d}t = \tfrac{4}{3}\pi a^3 \rho C_p \,\mathrm{d}T/\mathrm{d}t = -Q, \tag{7}$$

substituting (6) with $h = NuK_c/a$ gives

$$\mathrm{d}T/\mathrm{d}t + JT = JT_m(t), \tag{7a}$$

where $J = 3NuK/a^2$. Multiplying (7 *a*) throughout by the integrating factor e^{Jt} allows it to be integrated:

$$T = Je^{-Jt}\int e^{Jt} T_m(t)\,\mathrm{d}t. \tag{8}$$

The role of Nu is now evident: it is a measure of how seriously cooling is affected by wall rock convection, interfacial melting, crystallization, etc. The response of cooling to these effects can be handled by an appropriate choice of Nu. For a solid sphere motionless in a fluid $Nu = 1$ and $Pe = 0$, cooling is solely by conduction. Moving the body causes convection to be important and Nu increases proportionally with velocity. Internal magmatic convection similarly increases the internal Nu, yet since the bulk of the resistance to cooling comes from the wall rock, the external Nu is the rate controlling factor. Internal convection here is not a question of Rayleigh–Bénard stability. That is, the horizontal temperature gradients, cooler near the border and warmer in the core, experienced by the magma will force it to convect sufficiently rapidly to make the transfer of heat (i.e. Nu) to the border large. Hence magmatic cooling is dependent on the ability of wall rock to dispose of heat.

Although a good many experiments have measured Nu for a sphere, these are almost exclusively for a solid sphere (e.g. Bird *et al.* 1960, p. 409). Using a low Reynolds number boundary-layer approach and assuming the fluid velocity and stream function on the body *surface* to be similar to the Hadamard–Rybczynski (1911) result, Levich (1962, p. 408) derived Nu as a function of Pe for a liquid sphere:

$$Nu = 0.46(Pe)^{\frac{1}{2}}. \tag{9}$$

This result holds only for large (i.e. greater than about 10) values of Pe, and it has been found to compare favourably with the experimental results of Head & Hellums (1968) on heat transfer from liquid drops.

The final expression describing magmatic cooling is obtained from (8) by choosing a suitable function for the time variation of the wall rock temperature ($T_m(t)$). The geotherm chosen here primarily for mathematical convenience and its resemblance to an actual geotherm is $T_m(t) = T_0 \cos(\pi t/2t_0)$; where T_0 is the temperature in the source region, and t_0 is the total ascent time. The constant of integration is determined using the initial condition $T = T_0$ when $t = 0$. The result is

$$\frac{T}{T_0} = \left\{\left(\frac{J}{b}\right)^2 \cos(bt) + \frac{J}{b}\sin(bt) + e^{-Jt}\right\}\left\{\left(\frac{J}{b}\right)^2 + 1\right\}^{-1}, \tag{10}$$

where $J = 3NuK/a^2$ and $b = \pi/2t_0$. This solution is entirely equivalent to solving (8) with $T_m = T_0$ and then, through the use of Duhamel's theorem (see below), convolving this result with $T_m(t)$.

To use (10) an ascent velocity is chosen, Pe is found, and Nu is recovered from (9) and substituted into (10) and T/T_0 is then given as a function of the dimensionless time Kt/a^2. The choice of Pe is, however, not independent of Kt/a^2 which also gives the ascent time; they are connected through the ascent distance L. That is, let $Kt/a^2 = A$, then the ascent time is $t = Aa^2/K$, the ascent velocity is $V = LK/Aa^2$, and $Pe = L/Aa$ where the dimensionless parameter $L/a = n$ now appears. Substituting n for L/a and Kt/a^2 for A into the result for Pe changes (9) into

$$Nu = 0.46(na^2/Kt)^{\frac{1}{2}}. \tag{11}$$

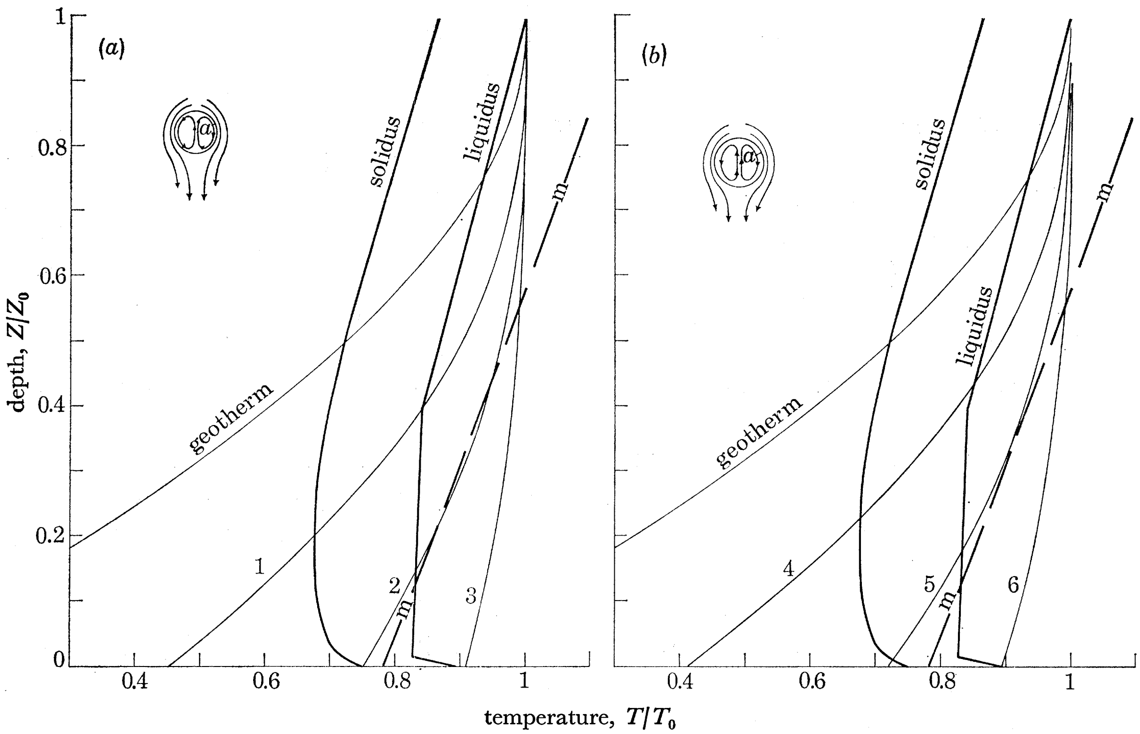

FIGURE 1. (a) Cooling by convection and conduction of a liquid sphere of magma of radius 0.8 km as it traverses lithosphere having a geotherm as shown. Labels are as given in figure 2. For curve 1, the ascent velocity (V) is *ca.* 10^{-6} m/s, the Fourier parameter (Kt/a^2) is 10^{-1}, the Nusselt number (Nu) is *ca.* 17.8 and the Peclet number (Pe) is *ca.* 1.5×10^3; for curve 2 these values are 10^{-5}, 10^{-2}, 56.2 and 1.5×10^4 respectively; and for curve 3, 10^{-4}, 10^{-3}, 178.2 and 1.5×10^5 respectively. Note the relatively large values of Pe, and compare these and V with those of (b) which is for a larger body. (b) Cooling by convection and conduction of a liquid sphere of magma of radius 6 km as it traverses lithosphere having a geotherm as shown. For curve 4, the ascent velocity is *ca.* 10^{-8} m/s, Kt/a^2 is 1.0, Nu is *ca.* 2.0 and Pe is *ca.* 9.5; for curve 5 these values are 10^{-7}, 10^{-1}, 6.5 and 10^2 respectively; and for curve 6, 10^{-6}, 10^{-2}, 20 and 10^3 respectively. See legend of figure 2 for other information.

Evidently any cooling curve calculated using (10) and (9) or (11) is completely characterized, for convective–conductive cooling, by choice of the dimensionless group Kt/na^2. In practice, however, L is the thickness of the lithosphere and any choice of n and ascent velocity or Kt/na^2 involves choosing a characteristic body size; the inherent connection between Pe and Kt/a^2 necessitates choosing a.

Cooling curves calculated using (10) and (11) are shown in figures 1*a* and *b*. (These curves were inadvertently calculated with the constant in (9) and (11) being 0.23. For the curves of interest here this small correction has not warranted recalculation.)

Elastic crack

Here the magma is quickly injected from one point to another; it is essentially continually thrust into contact with a medium at a lower temperature. The problem can be viewed as a body of initial temperature T_0 which is placed in contact with a medium (the wall rock) of temperature T_m. This problem for a semi-infinite plate has been considered by Williamson & Adams (1919), and from their solution the average temperature (T) at any time is given by

$$\frac{T-T_m}{T_0-T_m} = \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{\exp\{-(2m-1)^2\pi^2Kt/4a^2\}}{(2m-1)^2}. \tag{12}$$

This result is for a constant wall rock temperature, but it can be adapted to a variable wall rock temperature ($T_m(t)$) by convolving (12) with $T_m(t)$. This result is a good deal more cumbersome than the previous result and it has proven more convenient to use (12) in a simple numerical scheme. The wall rock temperature T_m is varied (temporally) from one location to another during ascent by solving (4) at n locations where the nondimensional time spent at each location is $1/n$ of the total ascent time (Kt/a^2) and the initial temperature at the nth location $T_{0,n}$ is given by the final temperature at the $n-1$ location (i.e. $T_{0,n} = T_{n-1}$). The use of (12) in this manner also allows great flexibility in choice of the functional form for the geothermal gradient. This scheme, however, is only stable when n is small (less than about 25), and these results are not as accurate as the previous ones.

The boundary condition used to derive (12) was that the boundary of the magma was instantly placed in contact with wall rock of temperature T_m. For a moving wall that is continually brought into contact with fresh, cool wall rock it is the most likely and accessible boundary condition to apply. This boundary condition certainly places an upper bound on how fast the magma cools and if the crack moves very fast this is sufficient. If heat is conducted ahead of the body, however, the magma will travel with a halo of heat (thermal boundary layer) about it. Lovering (1935), Ingersoll, Zobel & Ingersoll (1948) and Jaeger (1964) found for a body of magma placed instantly in contact with a cooler medium, but without holding the boundary temperature constant, that the boundary temperature quickly reaches a temperature which is the average of the two initial temperatures and this temperature persists for a large fraction of the total cooling time. Thus to relax the above boundary condition a contact temperature (at the nth position) which is the average of the magma (at the $n-1$ position) and the undisturbed wall rock temperature was also used. Cooling curves for these models are given in figure 2.

Pipe flow

Heat transfer between a hot fluid and a cooler pipe wall or surrounding medium has been studied extensively (e.g. McAdams 1954; Knudsen & Katz 1958; Roshenow & Choi 1961; Kays 1966; Morton 1960; Gupta 1973). Here the problem is more involved than either of the previous problems. When hot fluid enters a pipe and encounters a change in wall temperature, it begins to develop steady velocity and temperature profiles. The lengths of pipe encountered before these steady profiles develop are called, respectively, the hydrodynamic and thermal entry lengths; these lengths are a function of the Reynolds number and Prandtl number (Kays 1966, p. 119).

The Prandtl number, a ratio of momentum diffusivity to thermal diffusivity, here is large (*ca.* 10^4) implying that the magma velocity profile will be fully developed (i.e. reach steady state) much sooner than the temperature profile. Thus the velocity profile can be assumed fully developed upon entering the pipe. The thermal entry length depends on the boundary conditions, but in general it is proportional to $aRePr$, or Va^2/K where a is the pipe radius. For $V \sim 10^{-7}\,\mathrm{m\,s^{-1}}$, $a \sim 1\,\mathrm{km}$, and $K \sim 10^{-6}\,\mathrm{m^2\,s^{-1}}$, the thermal entry length is about 100 km. Since this is of the same order as the total ascent distance, the temperature profile will probably never

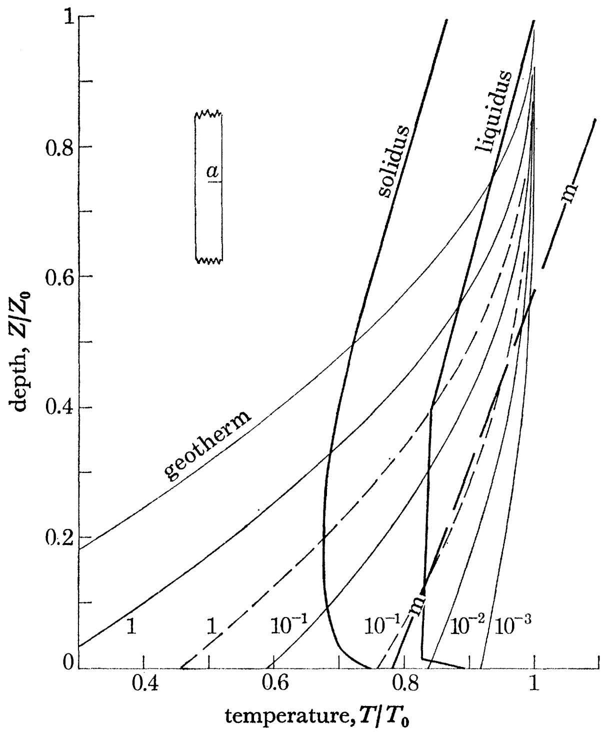

FIGURE 2. Cooling by conduction of a thin plate of andesitic magma as it traverses lithosphere possessing a geotherm as shown. From the dimensionless ascent time (Kt/a^2) given against each curve an ascent velocity can be calculated by choosing a value for a. The liquidus and solidus are for an andesitic magma containing about 1 % H_2O by mass (Green 1972), and dry mantle solidus, labelled m, is from Wyllie (1971). The cooling curves are for a magma initially containing no crystals; these curves do not apply below the liquidus. The solid curves represent cooling due to unpreheated wall rock, and the dashed curves are for preheated wall rock (but see text).

become fully developed. In addition, since the wall rock temperature (far from the pipe) continually changes with distance from the source, it is certain that no fully developed temperature profile will occur in magma flow in a pipe-like conduit traversing the lithosphere. Because of this unsteadiness no single Nusselt number can be used for the entire ascent, unlike in the case of a viscous sphere. Instead, the Nusselt number will be a function of distance from the source. Of course, if the magma moves up the pipe at 10^{-8} or $10^{-9}\,\mathrm{m\,s^{-1}}$, the thermal entry length is shorter. Velocities of this magnitude are considered unreasonably small.

Although a complete description of heat transfer from a fluid-filled pipe is impossible (but see Morton (1960) and Gupta (1973)), for simple cases certain interesting results can be found. Analytical solutions to heat transfer from fluid flow in a pipe can be obtained by specifying either a constant wall temperature or a constant heat flux along the pipe wall. And since the governing energy equation is linear and homogeneous, a sum of various solutions is also a solution. This property enables the constant wall temperature and the constant heat flux solutions to be used to obtain solutions with a variable boundary condition along the pipe. This amounts mathematically to convolving the initial solution with the desired function for the variable boundary condition. This procedure is also known as Duhamel's theorem (Carslaw & Jaeger 1959; Kays 1966). Because it is convenient to tie all calculations to the wall rock temperature, here the variable wall temperature problem seems more applicable; cooling outside the pipe is by conduction. The simple function $T_m = T_0 \cos(\pi x/2)$, where x is the fraction of the total ascent distance, is again chosen to describe the wall temperature; the details of the ensuing results hardly depend on the exact choice of this function, but for more complicated functions the resulting convolution integral is usually impossible to evaluate analytically.

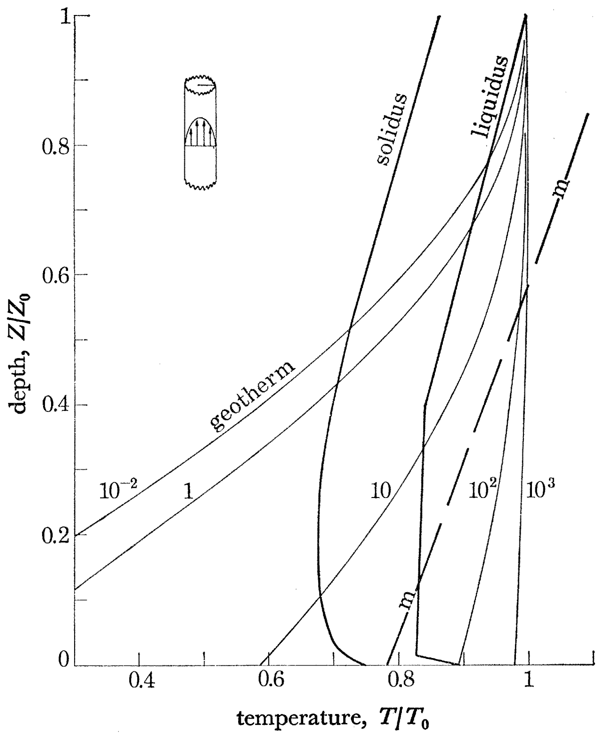

FIGURE 3. Cooling of magma as a function of ascent velocity or Peclet number (shown against the curves) as it flows up a pipe through lithosphere having a geotherm as shown. See legend of figure 2 for other labels.

The method of Kays (1966, chapter 8) is followed: his derivation is detailed and thorough and only a description of this method is of interest here. The energy equation is solved for hydrodynamically fully developed laminar flow with a parabolic (Poiseuille) velocity profile and with constant wall temperature beginning at the origin. This solution (Kays's eqns (8–33, 8–34)) is then convolved with the function describing the wall temperature. The mean magma temperature (T) as a function of distance from source, where the temperature is T_0, and ascent velocity is found to be

$$\frac{T-T_0}{T_0} = 8\sum_{n=0}^{\infty}\frac{G_n\lambda_n}{\lambda_n^4+Pe^2}\left[\cos\left(\tfrac{1}{2}\pi x\right) - 1 + \frac{Pe^2}{\lambda_n^2}\sin\left(\tfrac{1}{2}\pi x\right) + \frac{Pe^2}{\lambda_n^2}\{\exp\left(-\tfrac{1}{2}\lambda_n^2\pi x/Pe\right) - 1\}\right], \quad (13)$$

where Pe is the Peclet number (Va/K) and λ_n are the eigenvalues and G_n are numbers derived from the eigenfunctions obtained from the solution of a Sturm–Liouville-type equation associated with this problem. For completeness, when $n > 2$, $\lambda_n = 4n + \frac{8}{3}$, $G_n = 1.01276\,\lambda_n^{-\frac{1}{3}}$, and for $n = 0, 1, 2, \lambda^2 = 7.312$, 44.62, 113.8, and $G = 0.749$, 0.544, 0.463, respectively (Kays 1966, p. 125).

A set of cooling curves calculated by summing (13) numerically is presented in figure 3.

Discussion of cooling curves

In order to isolate the cooling effects dependent only on the heat transfer models the curves displayed do not contain the contribution to cooling from adiabatic expansion; if desired, this effect, which at $Z/Z_0 = 0$ can be shown to be $T/T_0 \approx 0.95$, can be added linearly. Strictly speaking, these curves do not apply at temperatures below the liquidus or for a magma initially containing crystals (but see below).

The shapes of the cooling curves are all quite similar. This is due to the shape of the geotherm. Several other functional forms for the geotherm were also used in the calculations and the resulting cooling curves differ little from those presented.

The effect of different boundary conditions is readily apparent in these figures. For a magma-filled crack moving past wall rock at a temperature equal to that of the ambient lithosphere cooling takes place about five times faster than when the wall rock is preheated by advancing magma. That is, for no preheating the magma must travel about five times faster if it is to reach the surface at the same temperature as a magma moving through preheated wall rock.

The Nusselt-number formulation for the viscous sphere moving through fluid wall rock offers the most complete description of heat transfer and these results are also probably the most accurate. These results involving two interrelated dimensionless groups dictate choice of body radius, and hence figures 1*a* and *b* show only curves for radii of 0.8 and 6 km.

A typical ascent velocity can only be extracted from any of these cooling curves if a characteristic body size is known. It is difficult, but obviously important, to determine a characteristic radius or volume for an average parcel of magma moving through the Earth. The only volumetric data available are obtained from extruded magma and the estimated volumes and shapes of exposed plutons. From the area containing many plutons mapped by Moore (1963) in the Sierra Nevada, Fyfe (1970) estimates an average pluton to have a volume equivalent to a sphere with a radius of from 3 to 7 km. Plutons in the central Aleutian Islands (e.g. Adak Island) have equivalent sphere radii of about 1–4 km. Like plutons, lava volume represents only a lower bound on the size of its associated magma chamber. Many volcanoes in the Aleutian Islands (see Coats 1950) seem to result from one, albeit sometimes long, period of volcanic activity, indicating perhaps that the size of a volcano may be a rough measure of the volume of its associated magma chamber. The volume of these composite and stratocone volcanoes can be approximated fairly well by a cone of radius equal to its height. The radius of a sphere having a volume equivalent to this cone is 0.625 of the volcano height. The largest Aleutian volcano has a summit elevation of about 3 km (Shishaldin) while more commonly the relief is about 1.75 km implying equivalent sphere radii of about 1–2 km which agrees roughly with the pluton result. In a detailed aeromagnetic study of some volcanoes and calderas in Japan, Muroi (1973) has interpreted the anomaly-causing bodies to have radii similar to those estimated for the Aleutian plutons. The radii associated with figures 1*a* and *b* were chosen as a result of these considerations.

If these volumes of magma are used to fill an ellipsoidal crack of thickness $2a_e$ and aspect ratio R the half-thickness of the crack is given by $a_e = a_s R^{-\frac{2}{3}}$, where a_s is the radius of a sphere of equivalent volume. Based on two-dimensional equilibrium profiles of cracks in an elastic medium typical values of R are from 10^3 to 10^4 (Weertman 1971*a*, *b*; Pollard & Muller 1976). Hence for equivalent volumes of magma the values of Kt/a^2 in figure 2 will be $R^{\frac{4}{3}}$ times *smaller* than those for a spherical body. If a magma-filled crack is to arrive at the surface at the same temperature as a ball of magma of equivalent volume it must travel $R^{\frac{4}{3}}$ times faster than the ball of magma. With all else being equal, for example, a crack with an aspect ratio of 10^3 must travel 10^4 times faster than a sphere of magma in order to arrive at the surface at the same temperature.

The results for flow of magma through a pipe (figure 3) are given in terms of the Peclet number. For values of Pe greater than about 5×10^3 ascent is nearly isothermal. If the erupted magma is to fit the petrologic characteristics stated already, aside from adiabatic cooling, Pe must be less than about 5×10^3. This sets an upper bound on the velocity; since the wall rock thermal diffusivity is about $10^{-6}\,\mathrm{m^2\,s^{-1}}$, $V < 5 \times 10^{-3}/a$ where a is the pipe radius. In the Aleutian Islands the volcanic centres – not the volcanoes – are on the order of 10 km in radius. Geologically reasonable conduit radii surely lie within the limits of 0.1 to 10 km, which give an upper bound of ascent velocities of 5×10^{-5} to $5 \times 10^{-7}\,\mathrm{m\,s^{-1}}$. Judging from the volume of lava and subcontinental crust which has accumulated in the central Aleutian Islands over the course of about 60 Ma, the larger velocity (smaller radius) is probably most applicable. The pipe model has the inherent disadvantage that for a 120 km column of magma the non-hydrostatic pressure developed relative to the higher density wall rock will cause the magma to flow at a large velocity. The non-hydrostatic pressure is given by $\Delta P = \Delta\rho g L$, where $\Delta\rho$ is the density contrast, g is gravitational acceleration and L is the ascent distance. For sensible values of $\Delta\rho$ and with L on the order of 100 km, ΔP is about 1 kbar or more, and since this is beyond the strength of most rocks, the pipe could not be capped. The ascent velocity at the centre of the pipe due to this pressure gradient is $V = (\Delta P/L)\, a^2/4\mu$, where a is the pipe radius and μ is the magma viscosity. Even for unfavourable values of a (1 m) and μ (10^4 P†), the ascent velocity is at least $1\,\mathrm{m\,s^{-1}}$, and the magma will ascend isothermally. Although near the surface (less than about 25 km) magma may be transported from a magma chamber to the volcano by flow similar to that in a pipe, it seems unlikely that magma can be transported from the Benioff zone to the surface solely by this means.

To estimate an ascent velocity cooling curves must be selected from those possible which fit with the observed petrologic characteristics of andesitic lavas. Aleutian and Tongan lavas are characterized in general by 1–50 % modal phenocrysts, a lack of high pressure phases ($P >$ *ca.* 10 kbar) indigenous to the magma, and a paucity of mantle-derived xenoliths (Marsh 1976*a*; Ewart 1976). The presence of phenocrysts deny that the magma was superheated just prior to eruption. The nature of the phenocrysts themselves reflect crystallization at low pressures and high temperature (i.e. $P \sim 5$ kbar, $T \sim 1100$–$1250\,^{\circ}\mathrm{C}$). If crystallization also took place at greater pressures, high-pressure phases should occasionally be found in the lavas. Thus a probable cooling curve might be one which strikes the liquids at a depth of less than about 20 km. If so, the magma is superheated over much of its ascent. Some evidence seems to support this possibility. Although rare, occurrences of mantle-derived olivine have been reported from one Aleutian lava (Marsh 1976*a*) and from the recent Tongan eruption on Metis Shoal (Ewart *et al.* 1973). These olivines are large, unzoned, sometimes slightly strained and of similar composition. Oddly enough, however, no coeval minerals (e.g. orthopyroxene, spinel, clinopyroxene, or garnet) have been found in these lavas. Since many cooling curves which strike the liquidus at depths of

† $1\ \mathrm{P} = 10^{-1}$ Pa s.

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0–20 km indicate that the magma is able to fuse the mantle, this singular presence of olivine, the mantle liquidus phase, may suggest that its coeval phases have been fused by the magma.

The presence of crystals in the initial magma holds the magma temperature near its liquidus until they are fused. The distance the magma must ascend before these crystals vanish by fusion depends on the ascent velocity (cooling path), heat of fusion, and volume fraction of crystals present. Nevertheless since the heat of fusion of these crystals is probably near that of diopside, 20.4 J g^{-1}, to melt 10 % of the crystals involves a loss of energy equivalent to a temperature decrease of about 25 K, assuming a specific heat of 0.1 J g^{-1} K^{-1}. Hence in figures 1 *a* and *b*, for example, for that group of cooling curves which strike the liquidus within about 25 km of the surface the magma will remain near its liquidus for about the first 10 % of its ascent. The result of crystals being present during the early stages of ascent is to move the cooling curves to lower temperatures; curves which in the standard state (i.e. no crystals) intersect the liquids at, say, 1 bar will, with 10 % crystals initially present, strike the liquidus at a few kilobars total pressure. The temperature will never rise above the liquidus if about 40 % crystals are initially in the magma and do not drop out during ascent.

Once the superheated magma crosses its liquidus crystals will begin forming and the heat of crystallization will create a heat source in the magma. In this region the magma may, depending on its ascent velocity, cool more slowly than the standard state cooling curves indicate. The importance of the heat derived from crystallization relative to that carried away by conduction is given by the last term in (4). This number is difficult to evaluate because e is so sensitive to the nucleation rate, growth rate, and solidification time. Yet if crystallization is to take place the magma temperature must drop with time, and thus this ratio must be less than unity. If crystallization takes place at a uniform rate the magma may cool only about half as fast as indicated by the cooling curves. This is because the addition of latent heat to the magma can be viewed as a doubling of the magma specific heat (e.g. Jaeger 1964) and twice as much heat must be carried away to cool by one degree than for the crystal-free magma.

These models of heat transfer are unrealistic in several ways. Magma is unlikely to ascend from source to volcano at a constant velocity. The common and sudden occurrence of near-surface earthquake swarms and harmonic tremor just prior to eruption of lava may signal a great increase in ascent velocity. Magma may also cool by losing a portion of itself through freezing near its margins and the formation of dikes and veins. And, since volcanic centres remain fixed for millions of years, the average temperature of the ascent path will surely increase with time. The uncertainties entered by these deficiencies are, however, probably well within the wide range of estimated ascent velocities. That is, since a characteristic ascent velocity for island-arc magma is completely unknown, an uncertainty in velocity of, say, two or even three orders of magnitude is surely acceptable for the present. Despite this inherent uncertainty, clearly for any body shape the velocity is relatively insensitive to the boundary conditions of heat transfer; widely different calculations (many others by the author are not presented here) give essentially similar results. This is not completely surprising since the basic feature of any boundary condition is the change in wall rock temperature by more than 1000 K degrees through the lithosphere. This undeniable condition dictates the results of any cooling model.

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Conclusions

A rather general, simple, and convenient formulation describing convective and conductive heat transfer from a magma moving through wall rock with a time-dependent temperature has been found (equation (8)). This formulation allows the effects of wall-rock fusion, magma crystallization, volatile loss, and any other mechanism of heat transfer or heat production to be treated through a judicious choice of the Nusselt number, *Nu*.

The aim in constructing purely phenomenological cooling models is to estimate an ascent velocity which can be used to investigate a dynamic model. Judging solely from the cooling curves, a spherical magma with a radius of 1 km must ascend at a velocity of at least 10^{-7} m s^{-1} to arrive at the surface without solidifying. A plate-shaped (crack) body of magma having the same volume as the sphere must, because of its large surface area, ascend at least 10^{-3} m s^{-1}. If andesitic magma ascends with the latter velocity, ultramafic xenoliths from the mantle should occasionally appear in the lavas, but these are rare. Hence the slower ascent velocities are more appropriate, but, conversely, the faster velocities may apply to the ascent of alkali-basalts. If magma begins its ascent with less than about 10 % crystals it will become superheated and remain so over much of its ascent. Superheating precludes changing the magma composition by crystal fractionation, and in the near-surface environment the magma is apt to be near its liquidus. Phenocrysts may typically form very near the surface which seems necessary from the preponderence of low-pressure phenocryst phases ubiquitous in andesitic lavas.

If the magma ascent is governed dynamically by an equation similar to Stokes's, because of the great viscosity of the lithosphere the slowness of the ensuing ascent velocity ensures solidification of the magma unless it is unreasonably large. To be useful, Stokes's relation calls for a constant wall-rock viscosity out to a distance of at least ten body radii from the magma, and this condition is surely not met here. Instead the magma dynamics may be more closely governed by the viscosity of the wall rock within a thin thermal halo about the magma. In a companion study, to appear elsewhere, the ascent velocity of a hot viscous sphere which moves by softening a thin rind of wall rock about it was found (some of the equations are given in Marsh 1977). These results show the ascent velocity can only be increased to approximate that estimated from cooling by partial fusion of the wall rock. About 30 % melting allows the magma to ascend at 10^{-7} m s^{-1}; this much melting probably takes place within about 50 km of the surface. Thus the ascent velocity is probably variable, and the earliest bodies of magma do not reach the surface. It is only with repeated passage through the same region of the lithosphere that andesitic magma can ascend as a viscous blob and reach the surface. Repeated use of such a chimney, which is the rule in island arcs, heats the lithosphere locally which insulates succeeding magmas allowing them to ascend more slowly and still reach the surface. A viscous blob of magma penetrating virgin lithosphere probably moves at about 10^{-9} to 10^{-8} m s^{-1}. After the chimney has been in use for a few million years the velocity may increase to about 10^{-7} to 10^{-6} m s^{-1}.

W. M. Elsasser, S. H. Davis, G. M. Homsy, and J. A. Marsh kindly helped in a number of ways. This work is supported by National Science Foundation Grant EAR-75-17617 to the Johns Hopkins University. Travel funds were graciously supplied by the National Aeronautics and Space Administration via grant NGS-5090 to the Johns Hopkins University.

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